Derivace funkce – příklady:

① Derivace násobku funkce a součtu funkcí:

„$(cf)' = cf'$" ($c \in \mathbb{R}$ konstanta) a „$(f+g)' = f' + g'$"

- $(x^3 + \cos x)' = (x^3)' + (\cos x)' \underset{(T)}{=} 3x^2 - \sin x$, $x \in \mathbb{R}$;
- $(2 \operatorname{arctg} x)' = 2 \cdot (\operatorname{arctg} x)' \underset{(T)}{=} 2 \cdot \frac{1}{1+x^2}$, $x \in \mathbb{R}$;
- $(x - 2\operatorname{arctg} x)' = (x)' + (-2)(\operatorname{arctg} x)' = 1 - 2 \cdot \frac{1}{1+x^2}$, $x \in \mathbb{R}$; $\left(= \frac{x^2-1}{x^2+1}\right)$

② Derivace součinu: $(f \cdot g)' = f' \cdot g + f \cdot g'$

- $(x^2 \ln x)' = (x^2)' \cdot \ln x + x^2 \cdot (\ln x)' \underset{(T)}{=} 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$, $x \in (0, +\infty)$;
- $(\sqrt{x} \cdot \sin x)' = (\sqrt{x})' \sin x + \sqrt{x} (\sin x)' \underset{(T)}{=} \frac{1}{2\sqrt{x}} \cdot \sin x + \sqrt{x} \cdot \cos x$;

! zde $\mathcal{D}f = \langle 0, +\infty)$, ale „zatím" jsme spočítali $f'(x)$ pro $x \in (0, +\infty)$

a (?) $f'(0+) \underset{def.}{=} \lim_{x \to 0+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0+} \frac{\sqrt{x} \sin x}{x} = „\frac{0}{0}" =$

$= \lim_{x \to 0+} \sqrt{x} \cdot \underbrace{\frac{\sin x}{x}}_{\to 1} = „0 \cdot 1" \underset{AL}{=} 1$, tj. $f'(0+) = 1$

a $\mathcal{D}f' = \langle 0, +\infty)$.

$(x^4 \cdot \ln(x) \cdot \operatorname{arctg} x)' = (x^4)' \cdot \ln x \cdot \operatorname{arctg} x + x^4 (\ln x)' \operatorname{arctg} x + x^4 \cdot \ln x \cdot (\operatorname{arctg} x)' = 4x^3 \cdot \ln x \cdot \operatorname{arctg} x + x^4 \cdot \frac{1}{x} \operatorname{arctg} x + x^4 \cdot \ln x \cdot \frac{1}{1+x^2}$, $x \in (0, +\infty)$

③ Derivace podílu: $\left(\frac{f}{g}\right)' = \frac{f'g - f\cdot g'}{g^2}$

- $\left(\frac{1-x}{1+x}\right)' = \frac{(1-x)'(1+x) - (1-x)(1+x)'}{(1+x)^2} \underset{(T)}{=} \frac{(-1)(1+x) - (1-x).1}{(1+x)^2} =$

$= \frac{-2}{(1+x)^2}, \quad x \neq -1;$

- $\left(\frac{x^3}{x^2-1}\right)' = \frac{(x^3)'(x^2-1) - x^3(x^2-1)'}{(x^2-1)^2} \underset{(T)}{=} \frac{3x^2(x^2-1) - x^3 2x}{(x^2-1)^2} =$

$= \frac{x^4 - 3x^2}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}, \quad x \neq \pm 1;$

- $\left(\frac{e^x}{3+x^2}\right)' = \frac{(e^x)'(3+x^2) - e^x(3+x^2)'}{(3+x^2)^2} = \frac{e^x(3+x^2) - e^x.2x}{(3+x^2)^2} =$

$= \frac{e^x(x^2-2x+3)}{(3+x^2)^2}, \quad x \in R;$

④ Derivace složené funkce $\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$

a) $\left(\sin(g(x))\right)' = \cos(g(x)) \cdot g'(x)$ $\quad \left((\sin y)' = \cos y\right)$

$(\sin(3x+2))' = \cos(3x+2).(3x+2)' = \cos(3x+2).3, \quad x \in R;$

$(\sin(3x^3))' = \cos(3x^3)\,(3x^3)' = 9x^2 \cos(3x^3), \quad x \in R;$

$\left(\sin\left(\frac{1}{x}\right)\right)' = \cos\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)' = -\frac{1}{x^2}\cdot\cos\left(\frac{1}{x}\right), \quad x \neq 0;$

$(\sin\sqrt{x})' = \cos(\sqrt{x}).(\sqrt{x})' = \frac{1}{2\sqrt{x}}\cdot\cos\sqrt{x}, \quad x > 0,$

ale zde opět, $D_f = \langle 0, +\infty)$, tedy, chybí "existence $f'(0+)$ (?):

$(\sin\sqrt{x})'_{x=0+} = \lim_{x\to 0+} \frac{\sin\sqrt{x}}{x} = "\frac{0}{0}" = \lim_{x\to 0+} \underbrace{\frac{\sin\sqrt{x}}{\sqrt{x}}}_{\to 1} \cdot \frac{1}{\sqrt{x}} \underset{T + AL}{=} "1 \cdot \frac{1}{0^+}" = +\infty!$

Tedy, zde, $D_{f'} = (0, +\infty)$

b) $\underline{\left(e^{g(x)}\right)' = e^{g(x)} \cdot g'(x)}$ $\quad \left((e^{y})' = e^{y},\ y \in R \right)$

!
- $\underline{\left(e^{-x}\right)'} = e^{-x} \cdot (-x)' = -e^{-x}, \ x \in R;$
- $\underline{\left(e^{x^2}\right)'} = e^{x^2} \cdot (x^2)' = 2x\, e^{x^2}, \ x \in R;$
- $\underline{\left(e^{\frac{1}{x}}\right)'} = e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)' = -\frac{1}{x^2} \cdot e^{\frac{1}{x}}, \ x \neq 0;$
- $\underline{\left(e^{\cos x}\right)'} = e^{\cos x} (\cos x)' = -\sin x \cdot e^{\cos x}, \ x \in R;$
- $\underline{\left(e^{\frac{x^2+1}{x^2-1}}\right)'} = e^{\frac{x^2+1}{x^2-1}} \left(\frac{x^2+1}{x^2-1}\right)' = e^{\frac{x^2+1}{x^2-1}} \dfrac{2x(x^2-1)-(x^2+1)\cdot 2x}{(x^2-1)^2} =$

$$= e^{\frac{x^2+1}{x^2-1}} \cdot \frac{-4x}{(x^2-1)^2}; \quad x \neq \pm 1;$$

- $\underline{\left(e^{\sqrt{x}}\right)'} = e^{\sqrt{x}} \cdot (\sqrt{x})' = \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}}$ pro $x \in (0, +\infty)$,

  i když $\underline{Df = \langle 0, +\infty)}$;

a $\underline{\left(e^{\sqrt{x}}\right)'_{x=0+}} \underset{def.}{=} \lim\limits_{x \to 0+} \dfrac{e^{\sqrt{x}} - e^{0}}{x} = \lim\limits_{x \to 0+} \dfrac{e^{\sqrt{x}} - 1}{x} = „\dfrac{0}{0}" =$

$$= \lim_{x \to 0+} \underbrace{\frac{e^{\sqrt{x}}-1}{\sqrt{x}}}_{\to 1 \text{ (T+VLSF)}} \cdot \frac{1}{\sqrt{x}} = „1 \cdot \frac{1}{0+}" \underset{AL}{=} +\infty,$$

tj. $\underline{Df' = (0, +\infty)}$

c) $\underline{\underset{(g(x)>0)}{\ln(g(x))} = \frac{1}{g(x)} \cdot g'(x) \left(= \frac{g'(x)}{g(x)}\right)}$, $\quad \left((\ln y)' = \frac{1}{y} \right)$

- $\underline{\ln(x^2+1)} = \frac{1}{x^2+1}(x^2+1)' = \frac{2x}{x^2+1}, \ x \in R$
- $\underline{\ln(\cos x)} = \frac{1}{\cos x} \cdot (\cos x)' = -\frac{\sin x}{\cos x}, \ x \in \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right),$

  $k \in Z$

d) $\underline{\left((g(x))^{\alpha}\right)' = \alpha (g(x))^{\alpha-1} \cdot g'(x)}$ · ($g(x)>0$ obecně)
(záleží na hodnotě α)

- $\underline{\left(\left(\frac{x+1}{x-1}\right)^2\right)'} = 2\left(\frac{x+1}{x-1}\right)\cdot\left(\frac{x+1}{x-1}\right)' = 2\left(\frac{x+1}{x-1}\right)\cdot\frac{1(x-1)-(x+1)\cdot 1}{(x-1)^2} =$

$$= \frac{-4(x+1)}{(x-1)^3}\ , \quad x \neq 1$$

- $\underline{\left(\sqrt{\frac{x-3}{x+2}}\right)'} = \left(\left(\frac{x-3}{x+2}\right)^{\frac{1}{2}}\right)' = \frac{1}{2}\left(\frac{x-3}{x+2}\right)^{-\frac{1}{2}}\cdot\left(\frac{x-3}{x+2}\right)' =$

$$= \frac{1}{2}\sqrt{\frac{x+2}{x-3}}\ \frac{x+2-(x-3)}{(x+2)^2} = \frac{5}{2}\sqrt{\frac{x+2}{x-3}}\ \frac{1}{(x+2)^2}$$

zde: $D_f = (-\infty,-2) \cup \langle 3,+\infty)$, ale $f'(x)$ máme jen v $(-\infty,-2)\cup(3,+\infty)$;
– (později budeme umět i vyšetřit $f'_+(3)$)

a obecněji: $\underline{\left(f(g(h(x)))\right)'} = f'(g(h(x)))\cdot(g(h(x)))' =$

$$\underline{= f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)}$$

- $\underline{\left(\sin\left(\sqrt{2-x}\right)\right)'} = \cos\left(\sqrt{2-x}\right)\cdot\left(\sqrt{2-x}\right)' =$

$$= \cos\left(\sqrt{2-x}\right)\cdot\frac{1}{2\sqrt{2-x}}\cdot(2-x)' =$$

$$= \cos\left(\sqrt{2-x}\right)\cdot\frac{-1}{2\sqrt{2-x}}\ , \quad x \in (-\infty,2)$$

- $\underline{\left(\ln(\ln(\ln x))\right)'} = \frac{1}{\ln(\ln x)}\left(\ln(\ln x)\right)' = \frac{1}{\ln(\ln x)}\cdot\frac{1}{\ln x}\left(\ln x\right)' =$

$$= \frac{1}{\ln(\ln(x))}\cdot\frac{1}{\ln x}\cdot\frac{1}{x}\ , \quad x > e\ ;$$

- $$\left(e^{\sqrt{\frac{1-x}{1+x}}}\right)' = e^{\sqrt{\frac{1-x}{1+x}}} \cdot \left(\sqrt{\frac{1-x}{1+x}}\right)' = e^{\sqrt{\frac{1-x}{1+x}}} \cdot \frac{1}{2\sqrt{\frac{1-x}{1+x}}} \cdot \left(\frac{1-x}{1+x}\right)' =$$

$$= e^{\sqrt{\frac{1-x}{1+x}}} \cdot \frac{1}{2}\sqrt{\frac{1+x}{1-x}} \cdot \frac{(-1)(1+x)-(1-x)}{(1+x)^2} =$$

$$= -e^{\sqrt{\frac{1-x}{1+x}}} \cdot \sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{(1+x)^2}, \quad x \in (-1,1)$$

(D_f je zde interval $D_f = (-1,1\rangle$, $f'_-(1)$ budeme „umět" později.)

A „vše dohromady" – další příklady výpočtu derivací:

- $$\underbrace{\left(x^3 \ln(\operatorname{arctg}(2x))\right)'}_{x>0} = (x^3)' \ln(\operatorname{arctg}(2x)) + x^3 \cdot \left(\ln(\operatorname{arctg}(2x))\right)' =$$

$$= 3x^2 \cdot \ln(\operatorname{arctg}(2x)) + x^3 \cdot \frac{1}{\operatorname{arctg}(2x)} \left(\operatorname{arctg}(2x)\right)' =$$

$$= 3x^2 \ln(\operatorname{arctg}(2x)) + \frac{x^3}{\operatorname{arctg}(2x)} \cdot \frac{1}{1+(2x)^2} \cdot 2$$

- $$\underbrace{\left(\ln\left(x+\sqrt{1+x^2}\right)\right)'}_{x+\sqrt{1+x^2}>0 \ \forall x,} = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(x+\sqrt{1+x^2}\right)' =$$

tj. $x \in R$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x\right) =$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

- $$\underset{(f(x)>0)}{\left(f(x)^{g(x)}\right)'} = \left(e^{g(x)\ln f(x)}\right)' = e^{g(x)\ln f(x)}\left(g(x)\ln f(x)\right)' =$$
$$= e^{g(x)\ln f(x)}\left(g'(x)\ln f(x) + g(x)\cdot\frac{f'(x)}{f(x)}\right)$$
$$\left(= f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right)\right)$$

- $$\underset{x\in\mathbb{R}}{\left((x^2+1)^{\sin x}\right)'} = \left(e^{\sin x\ln(x^2+1)}\right)' = (x^2+1)^{\sin x}\left(\sin x\cdot\ln(x^2+1)\right)' =$$
$$= (x^2+1)^{\sin x}\left(\cos x\cdot\ln(x^2+1) + \sin x\cdot\frac{2x}{x^2+1}\right)$$

- $$\underset{x\neq-1}{\left(\operatorname{arctg}\left(\frac{1-x}{1+x}\right)\right)'} = \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\cdot\left(\frac{1-x}{1+x}\right)' =$$
$$= \frac{1}{1+\frac{(1-x)^2}{(1+x)^2}}\cdot\frac{(-1)(1+x)-(1-x)\cdot 1}{(1+x)^2} =$$
$$= \frac{1}{(1+x)^2+(1-x)^2}(-2) = \frac{-2}{2(x^2+1)} = -\frac{1}{1+x^2}$$

- $$\underset{x\in\mathbb{R}}{\left(\frac{3}{(x^2+4)^3}\right)'} = \left(3(x^2+4)^{-3}\right)' = 3\left((x^2+4)^{-3}\right)' =$$
$$= 3\cdot(-3)\cdot(x^2+4)^{-4}(x^2+4)' = \frac{-18x}{(x^2+4)^4}$$

(lze derivovat jako složenou funkci „hned")